Accuracy of modal wave-front estimation from eye transverse aberration measurements

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ABSTRACT

The influence of random errors in measurement of eye transverse aberrations on the accuracy of reconstructing wave aberration as well as ametropia (mean power) and astigmatism parameters is investigated. The dependence of mentioned errors on a ratio between the number of measurement points and the number of polynomial coefficients is found for different pupil location of measurement points. Recommendations are proposed for setting these ratios.

Key words: wave-front deformation, eye transverse aberrations, Zernike polynomials, least-squares technique, astigmatism, errors

1. INTRODUCTION

Currently refractive surgery with custom cornea ablation is being rapidly extended. This technology provides an accurate and selectable cornea ablation and ensures the correction of not only ametropia (mean power) and astigmatism, but higher-order aberrations as well [1,2].

To achieve successful outcome of refractive corneal surgery, detailed preliminary examination of refractive performance of the eye's optical system is to be accomplished. Therefore, a new generation of refractometers, enabling to measure refraction or aberration characteristics of the eye at separate pupil points, is being developed [3–6]. One of spatially resolved refraction measurement techniques is based on measurement of a position of a thin laser beam at the retina plane [7]. This beam intersects the pupil in so called measurement points, i.e. pupil zones with areas, which are considerably smaller than area of the whole pupil. Once the eye's transverse aberrations have been measured, one can determine Zernike polynomials by using the least-squares technique. Zernike polynomials describe the wave-front deformation function (wave aberration) of the eye, which enables to evaluate to-be-ablated cornea shape and estimate important ophthalmic parameters of ametropia (mean power), astigmatism, and sight acuity.

Since the measurement of transverse aberration of the thin beam at the retina plane is inevitably accompanied by errors, and a total number of such measurement points is limited, there is a question about the accuracy of wave aberration and ophthalmic parameters estimation. For this reason, a purpose of this paper is to investigate quantitatively the influence of transverse aberration measurement errors and the number of Zernike modes on errors of reconstructing mentioned parameters.

2. METHOD

Zernike polynomials are widely used in optics to present the wave aberration function at pupil and object space coordinates [8]. If wave aberration and ophthalmic parameters of the eye are defined for one retinal point (at a fovea zone) and the eye's optical system has no axial symmetry relative to the visual axis, the expression for the wave aberration function may be written as follows [9, 10]:

$$W_j = W(\rho_j, \varphi_j) = \sum_n \sum_m R_{n,m}(\rho_j)\left[C_{n,m} \cdot cos\, m\varphi_j + S_{n,m} \cdot sin m\varphi_j\right], \qquad (1)$$

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where ρ_j, φ_j are polar coordinates of the *j*-th measurement point ($j=1..q$), q is a total number of measurement points, $C_{n,m}, S_{n,m}$ are Zernike polynomial coefficients, $R_{n,m}(\rho_j)$ are Zernike radial polynomial terms, calculated for the *j*-th measurement point according to formula[9, 10]:

$$R_{n,m}(\rho_j) = \sum_{k=0}^{\frac{1}{2}(n-m)} (-1)^k \frac{(n-k)! \cdot \rho_j^{n-2k}}{k! \left[\frac{1}{2}(n+m)-k\right]! \cdot \left[\frac{1}{2}(n-m)-k\right]!} . \quad (2)$$

where m and n are integers characterizing a type and an order of the wave aberration term, respectively, $n \geq m$, $n+m$ is even.

Total number *t* of terms in expression (1) (i.e. the number of Zernike coefficients) is determined by maximum values of the *n* and *m* indexes:

$$t = \frac{2nm - m^2 + 2n + 2m + z}{4} - z_o , \quad (3)$$

where z_0 is a number of coefficients, having no influence on the wave aberration value; $z_0=(n+4)/2$, if n_{max} is even; $z_0=(n+3)/2$, if n_{max} is odd; $z=4$, if m_{max} and n_{max} are even; $z=3$, if m_{max} and n_{max} are odd or m_{max} is odd and n_{max} is even; $z=2$, if m_{max} is even and n_{max} is odd.

The set of equations (1) for all (q) measurement points may be written in a matrix form as:

$$\mathbf{W} = \mathbf{L}\,\mathbf{C} . \quad (4)$$

where **W** is a vector, comprising q elements W_j, **L** is a rectangular matrix with dimensions $q \times t$, whose elements are values of Zernike polynomial terms, computed for all measurement points, **C** is a vector, consisting of t Zernike coefficients $C_{n,m}$ and $S_{n,m}$.

The objective of polynomial approximation of the wave aberration function is to determine the vector **C** elements. This may be accomplished with the least-squares technique according to known formula:

$$\mathbf{C} = (\mathbf{A}^{\mathbf{T}}\mathbf{E}\ \mathbf{A})^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{E}\ \mathbf{X} = \mathbf{B}\ \mathbf{X} , \quad (5)$$

where $\mathbf{B} = (\mathbf{A}^{\mathbf{T}}\mathbf{E}\ \mathbf{A})^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{E}$, **A** is a matrix with dimensions $2q \times t$, comprising coefficients a_{xc} , a_{xs} , a_{yc} , and a_{ys} from the following set of equations:

$$\Delta_x(\rho_j, \varphi_j) = \frac{f'}{n'} \cdot \sum_n \sum_m \left[a_{xc} \cdot C_{n,m} + a_{xs} \cdot S_{n,m}\right],$$

$$\Delta_y(\rho_j, \varphi_j) = \frac{f'}{n'} \cdot \sum_n \sum_m \left[a_{yc} \cdot C_{n,m} + a_{ys} \cdot S_{n,m}\right],$$

where

$$a_{xc} = \left.\frac{\partial R_{nm}(\rho)}{\partial \rho}\right|_{\rho=\rho_j} \cdot cos\,\varphi_j \cdot cos\,m\varphi_j + m\frac{R_{nm}(\rho_j)}{\rho_j} \cdot sin\varphi_j \cdot sin m\varphi_j ,$$

$$a_{xs} = \left.\frac{\partial R_{nm}(\rho)}{\partial \rho}\right|_{\rho=\rho_j} \cdot cos\,\varphi_j \cdot sin\,m\varphi_j - m\frac{R_{nm}(\rho_j)}{\rho_j} \cdot sin\,\varphi_j \cdot cos\,m\varphi_j\,,$$

$$a_{yc} = \left.\frac{\partial R_{nm}(\rho)}{\partial \rho}\right|_{\rho=\rho_j} \cdot sin\,\varphi_j \cdot cos\,m\varphi_j - m\frac{R_{nm}(\rho_j)}{\rho_j} \cdot cos\,\varphi_j \cdot sin\,m\varphi_j\,,$$

$$a_{ys} = \left.\frac{\partial R_{nm}(\rho)}{\partial \rho}\right|_{\rho=\rho_j} \cdot sin\,\varphi_j \cdot sin\,m\varphi_j + m\frac{R_{nm}(\rho_j)}{\rho_j} \cdot cos\,\varphi_j \cdot cos\,m\varphi_j\,,$$

where $\Delta_x(\rho_j,\varphi_j)$ and $\Delta_y(\rho_j,\varphi_j)$ are orthogonal projections of retinal transverse aberration for j-th measurement point on axes X and Y, respectively; **X** is a vector, comprising elements $\Delta_x(\rho_j,\varphi_j)$ and $\Delta_y(\rho_j,\varphi_j)$; **E** is a diagonal matrix of weighting coefficients, f' is the back focal length of the eye's optical system, n' is a refractive index of the vitreous humor. For the "standard" eye: $f' = 22.89$ mm, $n' = 1.337$.

According to expression (5), equation (4) may be rewritten as:

$$\mathbf{W} = \mathbf{L}\,(\mathbf{A}^{\mathbf{T}}\mathbf{E}\,\mathbf{A})^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{E}\,\mathbf{X}\,,$$

or

$$\mathbf{W} = \mathbf{HX}\,, \tag{6}$$

where $\mathbf{H} = \mathbf{L}\,(\mathbf{A}^{\mathbf{T}}\mathbf{E}\,\mathbf{A})^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{E} = \mathbf{L}\,\mathbf{B}$.

It may be shown, that values of ametropia A_D (as aberration of general defocusing) and primary astigmatism $|A'_s - A'_m|$ in diopters, and angle φ_{max} between a horizontal plane, passing through the visual axis, and a plane, in which a retinal spot size is the biggest due to astigmatism, can be found by means of correspondent Zernike coefficients as follows:

$$A_D = \frac{8\,C_{2,0}}{D}\,, \tag{7}$$

$$|A'_s - A'_m| = \frac{8}{D}\sqrt{C_{2,2}^2 + S_{2,2}^2}\,, \tag{8}$$

$$\varphi_{max} = \begin{cases} \frac{\pi}{4} - \frac{\beta}{2}, & \text{if } C_{2,0} > 0 \\ -\frac{\pi}{4} - \frac{\beta}{2}, & \text{if } C_{2,0} < 0 \end{cases}, \quad \beta = \text{arctg}\left(\frac{C_{2,2}}{S_{2,2}}\right) + \pi \cdot z_\beta\,, \quad z_\beta = 0, \pm 1, \tag{9}$$

where D is the entrance pupil diameter in mm, $C_{2,0}$, $C_{2,2}$, and $S_{2,2}$ are coefficients, computed in µm for normalized radial coordinates ρ_j of measurement points at the pupil plane. Here z_β must have the value, for which

$$sin\,\beta = \frac{C_{2,2}}{\sqrt{C_{2,2}^2 + S_{2,2}^2}}\,,\quad cos\,\beta = \frac{S_{2,2}}{\sqrt{C_{2,2}^2 + S_{2,2}^2}}\,.$$

Thus, to reconstruct the W_j values and compute the A_D, $|A'_s - A'_m|$, and φ_{max} parameters, the following steps are to be done:

1) set a grid of measurement points at the pupil plane;
2) provide the measurement of the $\Delta_x(\rho_j,\varphi_j)$ and $\Delta_y(\rho_j,\varphi_j)$ transverse aberrations at all pupil points and combine the **X** vector;

3) set number t of necessary Zernike coefficients $C_{n,m}$ and $S_{n,m}$;
4) compute Zernike coefficients according to equation (5);
5) estimate the values of function W as well as ametropia and astigmatism parameters using equations (4), (7)...(9).

Equations (5)...(9) indicate that accidental errors of estimating W_j, A_D, $|A'_s - A'_m|$, and φ_{max} depend on accidental measurement errors (i.e. errors of the vector **X** elements) as well as on structure and dimensions of the **A** and **H** matrices, given by the q and t numbers. To obtain these dependences, expressions for computing dispersions are to be found. According to the theory of functions with accidental arguments, dispersion of values of such a function equals to a sum of products of squares of its partial derivatives with respect to arguments and dispersions of arguments. It should be noted that this thesis is valid for uncorrelated arguments only.

According to equations (5) and (6), the **C** and **W** matrices are linear relatively to accidental arguments of the **X** vector. Therefore

$$\sigma_{C_i}^2 = \sum_{k=1}^{2q} B_{ik}^2 \sigma_{X_k}^2 , \tag{10}$$

$$\sigma_{W_j}^2 = \sum_{k=1}^{2q} H_{jk}^2 \sigma_{X_k}^2 . \tag{11}$$

where $\sigma_{X_k}^2$ is dispersion of the k-th element of the **X** vector.

Dispersions of A_D, $|A'_s - A'_m|$, and φ_{max} estimation errors may be found similarly:

$$\sigma_{A_D}^2 = \frac{64\, \sigma_{C_{2,0}}^2}{D^2} , \tag{12}$$

$$\sigma_{|A'_s - A'_m|}^2 = \frac{64}{D^2} \cdot \frac{\left(C_{2,2}^2 \cdot \sigma_{C_{2,2}}^2 + S_{2,2}^2 \cdot \sigma_{S_{2,2}}^2 \right)}{C_{2,2}^2 + S_{2,2}^2} , \tag{13}$$

$$\sigma_{\varphi_{max}}^2 = \frac{32}{D^2} \cdot \frac{\left(S_{2,2}^2 \cdot \sigma_{C_{2,2}}^2 + C_{2,2}^2 \cdot \sigma_{S_{2,2}}^2 \right)}{\left(C_{2,2}^2 + S_{2,2}^2 \right)^2} . \tag{14}$$

where $\sigma_{C_{2,0}}$, $\sigma_{C_{2,2}}$, and $\sigma_{S_{2,2}}$ are standard deviations of estimation errors of Zernike coefficients $C_{2,0}$, $C_{2,2}$, and $S_{2,2}$, respectively. Equations (1)...(6) and (10)...(14) form the basis, on which further investigations are carried out.

3. RESULTS

To provide numerical research of wave aberration and ophthalmic parameters estimation errors according to equations (10)...(14) and (1)...(6), a special computer program has been developed. It enabled to set an arbitrary location of measurement points at the pupil plane and a desirable number of Zernike modes.

Nine different grids of three types have been chosen for the analysis (fig. 1). The first type (grid #1, 4, and 7) has a radial distribution of measurement points. The grids of the second type (grids #2, 5, and 8) are generated to have approximately the same distances between neighboring measurement points. Finally, the grids of the third type (grids #3, 6, and 9) have an equal step along the axes of the orthogonal coordinate system. The choice of such distributions takes into account peculiarities of optical deflectors action as well as a construction of micro-lens arrays in Hartmann-Shack sensors.

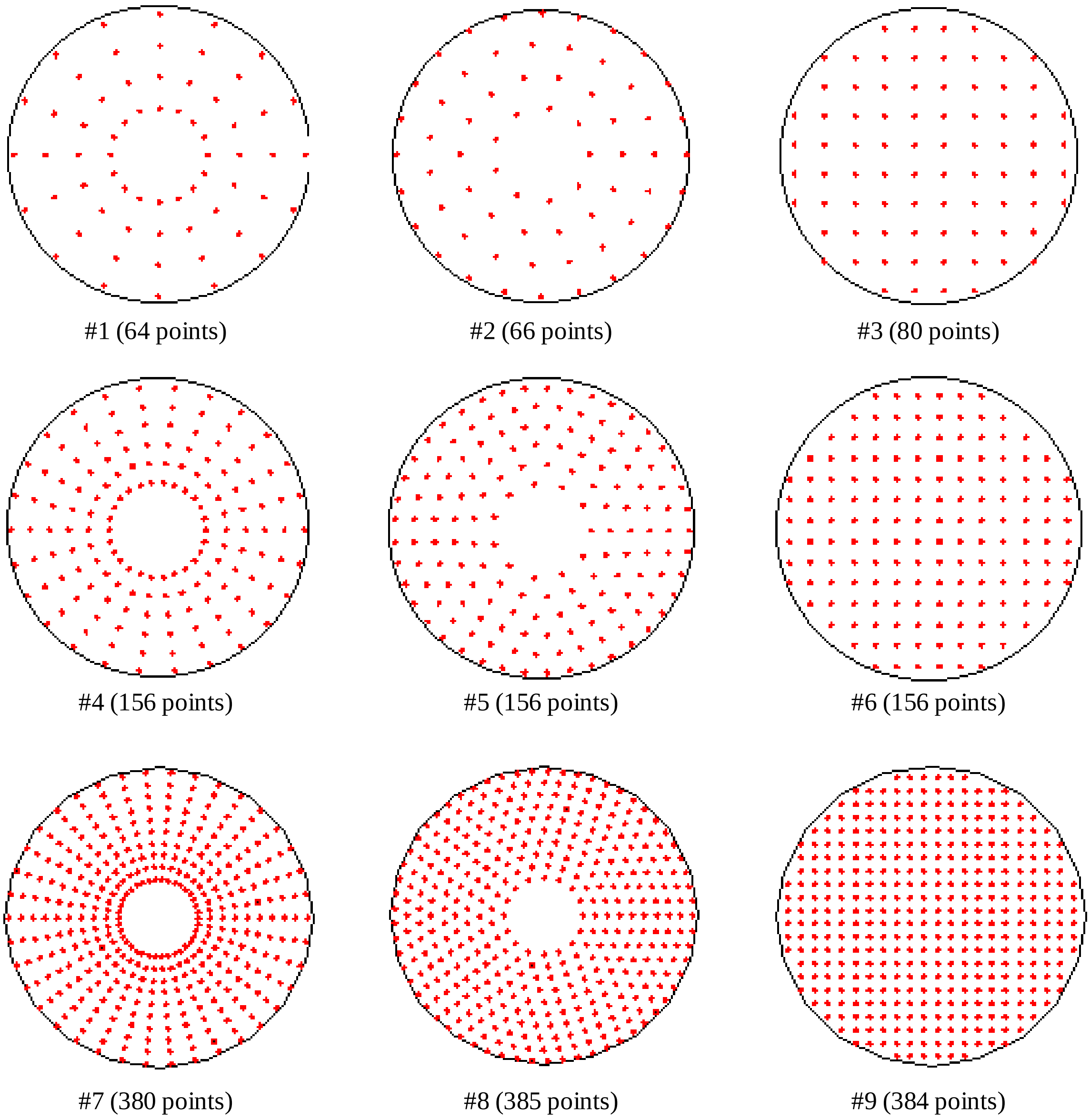
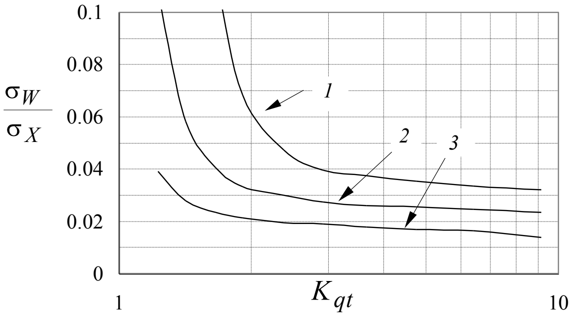


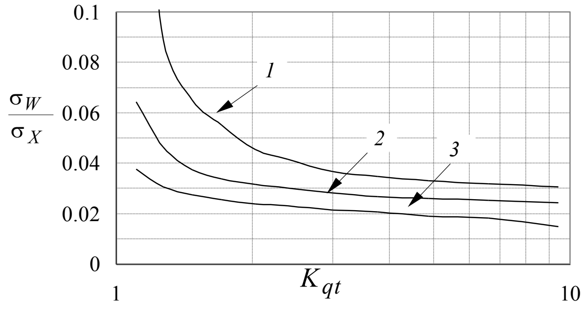
Fig. 1 Location of measurement points at the entrance pupil plane

When using equations (5)...(14), the experimentally proved fact of independence of $\sigma^2_{X_k}$ on index *k* was taken into account. It was established that values $\sigma^2_{X_k}$ are defined by electric noise of the photo-detective device, which practically does not depend on pupil coordinates of the measurement point. For this reason, the $\sigma^2_{X_k} = \sigma^2_X$ parameter as a constant was put out of the sum, and dispersion values of W_j, A_D, $|A'_s - A'_m|$, and φ_{max} were normalized on it.

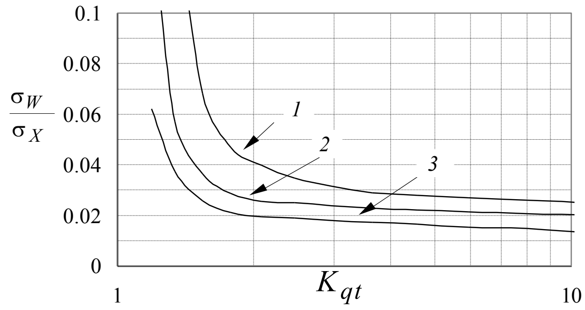
The results have confirmed that dispersions of W_j are constant for measurement points, located at the same distance from the pupil center. Therefore, relative standard deviations of W_j (indicated as σ_W / σ_X) as functions of factor $K_{qt} = \frac{2q}{t}$ are shown in fig. 2 for three different radii R. Radius R = 3 mm corresponds to the edge of the pupil zone. All figures are obtained with the help of computation subroutines having double precision.



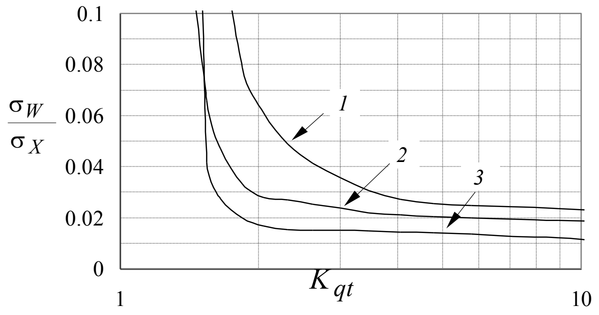
#1:
1 – R = 1 mm; *2* – R = 2.3 mm; *3* – R = 3 mm



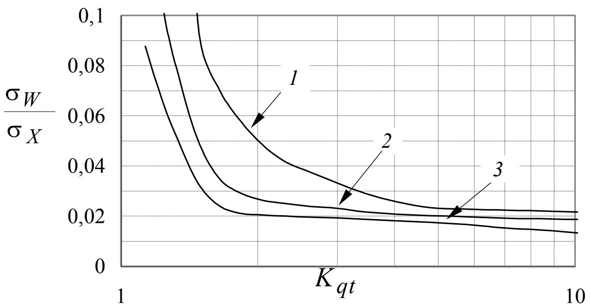
#2
1 – R = 1 mm; *2* – R = 2.3 mm; *3* – R = 3 mm



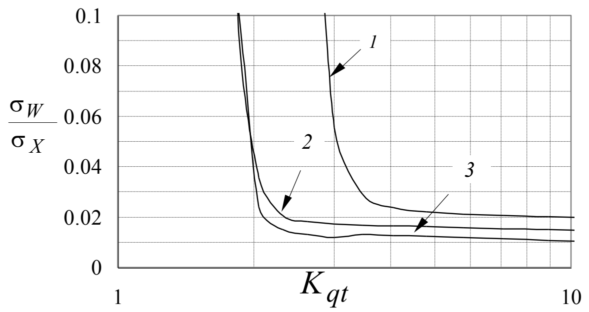
#3
1 – R = 1 mm; *2* – R = 2.1 mm; *3* – R = 2.7 mm



#4
1 – R = 1 mm; *2* – R = 2.6 mm; *3* – R = 3 mm

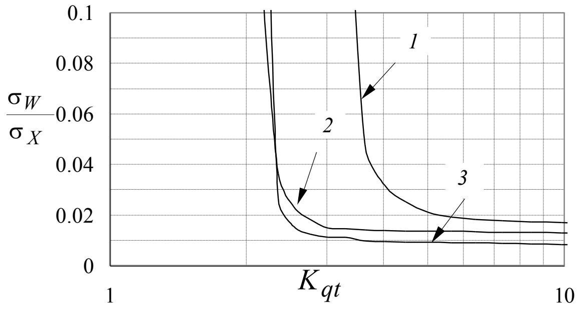


#5
1 – R = 1 mm; *2* – R = 2.6 mm; *3* – R = 3 mm



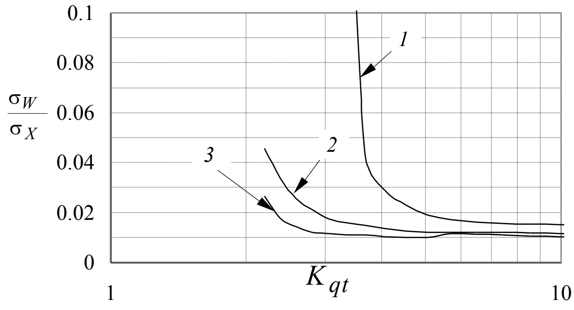
#6
1 – R = 0.3 mm; *2* – R = 1.9 mm; *3* – R = 2.8 mm

Fig. 2 Charts of wave aberration estimation errors



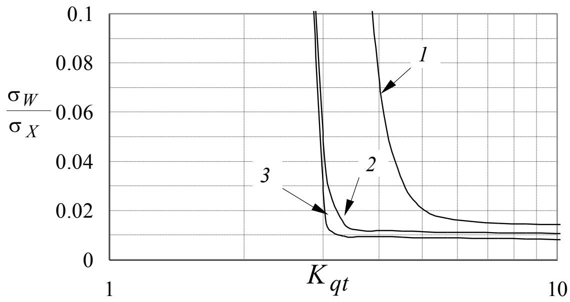
#7

1 – R = 0.8 mm; *2* – R = 2.5 mm; *3* – R = 3 mm



#8

1 – R = 0.8 mm; *2* – R = 2.3 mm; *3* – R = 3 mm



#9

1 – R = 0.4 mm; *2* – R = 2.1 mm; *3* – R = 2.9 mm

Continuation of fig. 2

According to equation (5), relative standard deviations σ_A / σ_X and $\sigma_{|A'_S - A'_M|} / \sigma_X$ depend preliminary on number q of measurement points (fig. 3).

The analysis of obtained results enabled to establish:

1) In general, relative error σ_W / σ_X essentially depends on a number and a pupil distribution of measurement points. However, if factor K_{qt} equals to 5...10, then mentioned error weakly correlates with points location and belongs to interval 0.04...0.01;
2) Independently on points location, relative error σ_W / σ_X quickly increases, if factor K_{qt} tends to unity;
3) Pupil points distribution does not influence on value σ_W / σ_X for q = 64...80 when $K_{qt} > 3$, for q = 156 when $K_{qt} > 4$, and for q = 380...385 when $K_{qt} > 5$;
4) Relative errors σ_A / σ_X and $\sigma_{|A'_S - A'_M|} / \sigma_X$ practically do not depend on pupil points location and are defined by the total number of measurement points.
5) Relative error $\sigma_{\varphi max} / \sigma_X$ has the same behavior as errors σ_A / σ_X and $\sigma_{|A'_S - A'_M|} / \sigma_X$, but, in addition, it decreases, when the astigmatism value increases.

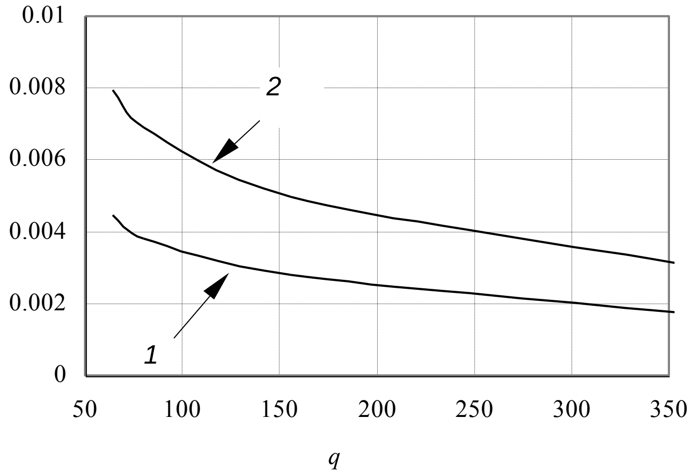


Fig. 3 Charts of ametropia and primary astigmatism estimation errors:

$$1 - \frac{\sigma_A}{\sigma_X}\left[\frac{\text{diopters}}{\mu\text{m}}\right],\ 2 - \frac{\sigma_{|A'_S - A'_M|}}{\sigma_X}\left[\frac{\text{diopters}}{\mu\text{m}}\right]$$

4. CONCLUSIONS

Presented materials and results of carried out investigations enable to estimate the potential accuracy of reconstructing wave aberration and ophthalmic parameters, if functional parameters of a spatially resolved refractometer are known. On the other hand, they enable to determine principle features of wave-front reconstruction, if required accuracy of wave aberration or ophthalmic parameters estimation is given.

To obtain a practical independence of wave aberration estimation errors on pupil location of measurement points, the total number of measurement points should at least 2.5 times be greater than the number of polynomial terms (number of Zernike coefficients).

REFERENCES

1. J. Schwiegerling and R. B. Snyder, "Custom photorefractive keratectomy ablations for the correction of spherical and cylindrical refractive error and higher-order aberration," *J. Opt. Soc. Am. A*, **15**. pp. 2572–2579, 1998.
2. S. Klein, "Optimal corneal ablation for eyes with arbitrary Hartmann-Shack aberrations," *J. Opt. Soc. Am. A*, **15**, pp. 2580–2586, 1998.
3. R. H Webb., C. M Penney., and K. D. Thompson, "Measurement of ocular local wavefront distortion with a spatially resolved refractometer," *Applied Optics*, **31**, pp. 3678–3686, 1992.
4. J. Liang., B. Grimm, S. Goelz, and J. F. Bille,."Objective measurement of wave aberrations of the human eye with the use of a Hartmann-Shack wave-front sensor," *J. Opt. Soc. Am. A.*, **11**, pp. 1949–1957, 1994.
5. J. Liang, D. R. Williams, and D. T. Miller, "Supernormal vision and high-resolution retinal imaging through adaptive optics," *J. Opt. Soc. Am. A.*, **14**, pp. 2884–2892, 1997.
6. G. Walsh, M. J. Cox, "A new computerized video-aberroscope for the determination of aberration of the human eye," Ophthalmic Physiol Opt., **15**, pp. 403–408, 1995.
7. V. V. Molebny, I. G. Pallikaris, L. P. Naoumidis, I. H. Chyzh, S. V. Molebny, V. M. Sokurenko, "Retina ray-tracing technique for eye-refraction mapping," *Proc. of SPIE*, **2971**, pp. 175–183, 1997.
8. I. W. Kwee and J. M. Braat, "Double Zernike expansion of the optical aberration function," *Pure Applied Optics*, **2**, pp. 21–32, 1993.
9. M. Born, E. Wolf, *Principles of optics*, Pergamon, New York, 1983.
10. S. N. Bezdidko, "The use of Zernike polynomials in optics," Sov. J. Opt. Technol. **41**, pp. 425–429, 1974.