Eye aberrations analysis with Zernike polynomials

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ABSTRACT

New horizons for accurate photorefractive sight correction, afforded by novel flying spot technologies, require adequate measurements of photorefractive properties of an eye. Proposed techniques of eye refraction mapping present results of measurements for finite number of points of eye aperture, requiring to approximate these data by 3D surface. A technique of wave front approximation with Zernike polynomials is described, using optimization of the number of polynomial coefficients. Criterion of optimization is the nearest proximity of the resulted continuous surface to the values calculated for given discrete points. Methodology includes statistical evaluation of minimal root mean square deviation (RMSD) of transverse aberrations, in particular, varying consecutively the values of maximal coefficient indices of Zernike polynomials, recalculating the coefficients, and computing the value of RMSD. Optimization is finished at minimal value of RMSD. Formulas are given for computing ametropia, size of the spot of light on retina, caused by spherical aberration, coma, and astigmatism. Results are illustrated by experimental data, that could be of interest for other applications, where detailed evaluation of eye parameters is needed.

Keywords: eye aberrations, sight correction, photorefractive operations, eye refraction map, refraction non-homogeneity, Zernike polynomials, eye refraction measurement

1. INTRODUCTION

At early stages of photorefractive keratectomy, the problem of sight correction seemed to be quite transparent: changing the radius of cornea curvature would solve the problem. Later, real measured shape of patient's cornea being non-spherical 3D function was involved into consideration[1 – 3]. Ablation procedure should transform it to spherical shape. The problem could be solved with flying-spot technology. But it turns out that declinations of the cornea shape from sphericity are not the only refractive-origin causes of low sight acuity. Non-homogeneous refraction distribution inside the eye makes another sensible contribution to spatial variance of the eye's focal power. Several techniques are proposed for eye refraction mapping[4 – 8].

Information on local eye aberrations is got usually in finite number of points of eye aperture. Therefore, to get a map of the to-be-ablated profile, suitable for ablation procedures, one must transform the discrete set of values into a continuous surface. The criterion of the quality of such transform could be the nearest proximity of the resulted continuos surface to the values calculated for the given discrete points. In this work, we consider the methodology of approximation using optimization of the length of Zernike polynomials.

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2. OPTIMIZATION OF POLYNOMIAL'S LENGTH

In general case, optical system of a real eye is axially asymmetric, therefore decomposition of its wave aberration function $W_{nm}(\rho,\varphi)$ contains sine terms, that take into account asymmetry of wave-front deformations[9, 10]:

$$W_{nm}(\rho,\varphi) = \sum_{n}\sum_{m} R_n^m(\rho)\cdot\left[C_{ynm}\cdot\cos(m\varphi) + C_{xnm}\cdot\sin(m\varphi)\right], \tag{1}$$

where ρ, φ are polar coordinates in the plane of eye's pupil; ρ is normalized from 0 to 1; C_{ynm}, C_{xnm} are coefficients of Zernike polynomials;

$$R_n^m(\rho) = \sum_{k=0}^{\frac{1}{2}(n-m)} (-1)^k \frac{(n-k)!\cdot\rho^{n-2k}}{k!\cdot\left[\frac{1}{2}(n+m)-k\right]!\cdot\left[\frac{1}{2}(n-m)-k\right]!},$$

$n \geq m$, $n+m$ being even.

Relationship between wave-front deformation function and transverse aberrations $\delta y'(\rho,\varphi)$ and $\delta x'(\rho,\varphi)$ of an eye looks as follows:

$$\delta y'(\rho,\varphi) = \left[\cos\varphi\cdot\frac{\partial W(\rho,\varphi)}{\partial\rho} - \frac{\sin\varphi}{\rho}\cdot\frac{\partial W(\rho,\varphi)}{\partial\varphi}\right], \tag{2}$$

$$\delta x'(\rho,\varphi) = \left[\sin\varphi\cdot\frac{\partial W(\rho,\varphi)}{\partial\rho} + \frac{\cos\varphi}{\rho}\cdot\frac{\partial W(\rho,\varphi)}{\partial\varphi}\right]. \tag{3}$$

Evaluation of C_{ynm} and C_{xnm}, based on measurements of transverse aberrations at different scan sites, may be accomplished by the least-squares method (LSM) [11, 12]. The solution is presented usually in matrix form:

$$\mathbf{C} = (\mathbf{A}^{\mathrm{T}}\cdot\mathbf{K}\cdot\mathbf{A})^{-1}\,\mathbf{A}^{\mathrm{T}}\cdot\mathbf{K}\cdot\mathbf{Y_o}, \tag{4}$$

where **C** is a column vector of unknown (to-be-calculated) Zernike coefficients; **A** is a matrix, consisting of derivatives of Zernike polynomials at given sites of approximation; **K** is a matrix of weight factors; $\mathbf{Y_o}$ is a column vector of transverse aberrations for two directions.

Total number of equations in the expression (4) equals to doubled number of approximation sites. The number of unknown Zernike coefficients N_C can be obtained as:

$$N_c = \frac{2nm - m^2 + 2n + 2m + z}{4} - z_o, \tag{5}$$

where $z=4$, if both m and n are even; $z=3$, if both m and n are odd or m is odd and n is even; $z=2$, if m is even and n is odd; n, m are maximal indices of Zernike polynomials; z_0 is a number of "non-significant" Zernike coefficients, i. e., coefficients, that do not influence the function of wave-front deformation; $z_0=(n+4)/2$, if n is even, and $z_0=(n+3)/2$, if n is odd.

Using the LSM is advisable because of the fact that coefficient estimates are non-displaced ones irrespectively of error distribution. Besides, according to the Gauss-Markov theorem, LSM gives the most accurate estimation of coefficients among the class of estimations, which are non-displaced and represent linear combination of initial data[12].

Accuracy of approximation depends on quantity of Zernike coefficients. According to the criterion of minimization of signal/noise ratio[13, 14], the number of modes must be reduced. However, the function of wave-front deformation of a real eye has rather complex character and can contain some local hills[15, 16]. That is why, it is not reasonable to reconstruct it with the degree of Zernike polynomials, smaller than 2.

On the other hand, because of non-orthogonality of the derivatives of Zernike polynomials, cross-coupling occurs when number of modes increases. Hence, excessive increase in polynomials degree would result inevitably in poor conditionality of matrix of normal equations.

Thus, there are some critical maximal indices *m* and *n*, corresponding to optimal approximation of initial data (in the meaning of the best coincidence of the approximated function with the initial data). Optimal values of the *m* and *n* can not be found analytically. Therefore, we use numerical method based on statistical estimates.

The important property of LSM, that can be used for evaluation of optimal *m* and *n*, is that the estimate of dispersion D_R of transverse aberrations can be obtained irrespectively of the kind of error distribution[12]:

$$D_R = \frac{R}{N_E - N_C}, \tag{6}$$

where R is residual sum of squares of deviations of estimated transverse aberration values from initial values; $R=\mathbf{V}^\mathbf{T}\cdot\mathbf{K}\cdot\mathbf{V}$, where $\mathbf{V}=\mathbf{Y}-\mathbf{Y_0}$ is a column vector of residuals; $\mathbf{Y}=\mathbf{A}\cdot\mathbf{C}$ is a column vector of estimated transverse aberration values at approximation sites; N_E is the number of equations; N_C is the number of unknown coefficients.

Taking into account the above mentioned considerations, we propose to use the criterion of minimal root mean square deviation (RMSD) of transverse aberrations:

$$\sigma = \sqrt{D_R} = \sqrt{\frac{R}{N_E - N_C}}. \tag{7}$$

The procedure consists in varying consecutively the values of maximal indices *m* and *n*, recalculating Zernike coefficients, and computing the value of RMSD. The "best" indices *m* and *n* are determined having minimal value of RMSD.

3. EVALUATION OF OPHTHALMOLOGIC PARAMETERS

Asymmetry of eye's optical system can produce all types of primary aberrations, first of all, focus shift (n=2, m=0), spherical aberration ($n\geq4$, m=0), coma ($n\geq3$, m=1), and astigmatism ($n\geq2$, m=2). It is known that small values of *n* correspond to primary aberrations. According to (1)…(3), primary transverse aberrations of the beam on retina can be found as follows:

- focus shift (n=2, m=0):

$$\begin{aligned}\delta y'(\rho,\varphi) &= 4\cdot C_{y20}\cdot\rho\cdot\cos(\varphi);\\ \delta x'(\rho,\varphi) &= 4\cdot C_{y20}\cdot\rho\cdot\sin(\varphi);\end{aligned} \tag{8}$$

- 3rd order spherical aberration (n=4, m=0):

$$\begin{aligned}\delta y'(\rho,\varphi) &= 12\cdot C_{y40}\cdot\rho\cdot(2\rho^2-1)\cdot\cos(\varphi);\\ \delta x'(\rho,\varphi) &= 12\cdot C_{y40}\cdot\rho\cdot(2\rho^2-1)\cdot\sin(\varphi);\end{aligned} \tag{9}$$

- 3rd order coma (n=3, m=1):

$$\begin{aligned}\delta y'(\rho,\varphi) &= 2\cdot C_{y31}\cdot(3\rho^2-1)+3\cdot\rho^2\cdot\sqrt{C_{y31}^2+C_{x31}^2}\cdot\sin(\alpha_c+2\varphi);\\ \delta x'(\rho,\varphi) &= 2\cdot C_{x31}\cdot(3\rho^2-1)-3\cdot\rho^2\cdot\sqrt{C_{y31}^2+C_{x31}^2}\cdot\cos(\alpha_c+2\varphi);\end{aligned} \tag{10}$$

where

$$\alpha_c = \mathrm{arctg}\,\left(\frac{C_{y31}}{C_{x31}}\right)+\pi\cdot q\,,\quad q=0,\pm1, \tag{11}$$

suggesting q, for which

$$\sin(\alpha_c)=\frac{C_{y31}}{\sqrt{C_{y31}^2+C_{x31}^2}}\,,\ \cos(\alpha_c)=\frac{C_{x31}}{\sqrt{C_{y31}^2+C_{x31}^2}}\,;$$

- 3rd order astigmatism (n=2, m=2):

$$\begin{aligned}\delta y'(\rho,\varphi) &= 2\cdot\rho\cdot\left[C_{y22}\cdot\cos(\varphi)+C_{x22}\cdot\sin(\varphi)\right];\\ \delta x'(\rho,\varphi) &= 2\cdot\rho\cdot\left[-C_{y22}\cdot\sin(\varphi)+C_{x22}\cdot\cos(\varphi)\right].\end{aligned} \tag{12}$$

Functions $\delta y'(\rho,\varphi)$ and $\delta x'(\rho,\varphi)$ allow to compute the dimensions of light spot on retina, the shape of its edge, as well as other parameters and characteristics. In further considerations, we use also the function of transverse aberration of a thin beam on retina:

$$\delta(\rho,\varphi)=\sqrt{\delta y'^2(\rho,\varphi)+\delta x'^2(\rho,\varphi)}\,. \tag{13}$$

3.1. Ametropia

When ametropia occurs, beams entering the eye in parallel to visual axis, are not focused on retina, i.e. they form defocused image of the point in infinity. In this case, $C_{y20}\neq 0$. Substituting (8) into (13), one obtains

$$\delta(\rho,\varphi)=4\cdot\rho\cdot C_{y20}\,, \tag{14}$$

$\delta(1,\varphi)=4\cdot C_{y20}$ at $\rho=1$.

Thus, the spot on retina is circle-shaped having diameter $2\cdot r_d=8\cdot C_{y20}$. Distance between retina and rear focus of an eye is

$$z'=-\frac{2\cdot\delta(1,\varphi)\cdot f'}{D}=-\frac{8\cdot C_{y20}\cdot f'}{D}$$

where f' is rear focal length of an eye, D is the diameter of pupil scan zone.

According to the Newton's formula, the far point of clear sight (i. e., the axial point optically conjugated with retina) is situated at a distance

$$z = \frac{f \cdot f'}{z'} = -\frac{f \cdot D}{8 \cdot C_{y20}}$$

from front focus. Here, f is the front focal length of an eye.
As the distance $z >> |f|$, one can obtain the value of ametropia

$$A \cong \frac{1000}{z} = -\frac{8000 \cdot C_{y20}}{f \cdot D},$$

or

$$A = \frac{8000 \cdot n' \cdot C_{y20}}{f' \cdot D} \text{ [diopters]}, \tag{15}$$

where n=1.337. Hence, coefficient C_{y20} testifies not only about the presence of ametropia, but allows computing its value.

3.2. 3rd order spherical aberration

Presence of the 3rd order spherical aberration is defined by the coefficient C_{y40}. If $C_{y40} \neq 0$, spherical aberration appears, influencing on sight acuity. This influence can be evaluated by the dimension of light spot on retina. Substituting (9) in (13), we obtain $\delta(\rho, \varphi) = 12 \cdot \rho \cdot (2\rho^2 - 1) \cdot C_{y40}$, i.e. the spot has a round shape with diameter

$$2 \cdot r_s = 2 \cdot \delta(\rho, \varphi)\big|_{\rho=1} = 24 \cdot C_{y40}. \tag{16}$$

A zone of object space, optically conjugated with this spot, has angular aperture

$$\theta = \frac{24 \cdot C_{y40}}{|f|} \text{ [radians]}. \tag{17}$$

3.3. 3rd order coma

It is evident from expressions (10), that for ρ=0, the initial point of the coma spot is displaced from coordinate origin on distance $\delta y'(0,0) = -2 \cdot C_{y31}$ and $\delta x'(0,0) = -2 \cdot C_{x31}$. As far as coma's head contains large part of light energy, it changes angular position of image centroid on retina. Maximal axial and lateral dimensions of the coma spot are determined as related to the axis of spot symmetry at ρ=1 using formulas:

- for axial size:

$$\delta l_c' = 9 \cdot \sqrt{C_{y31}^2 + C_{x31}^2}, \tag{18}$$

- for lateral size:

$$\delta r_c' = 6 \cdot \sqrt{C_{y31}^2 + C_{x31}^2}. \tag{19}$$

3.4. 3rd order astigmatism

By substituting (12) into (13), one can obtain function $\delta(\rho, \varphi)$:

$$\delta(\rho,\varphi) = 2\cdot\rho\cdot\sqrt{S + Q\cdot\sin(\beta + 2\varphi)}\ , \tag{20}$$

where

$$S = C_{y22}^2 + C_{x22}^2 + 4C_{y20}^2\ ;\quad Q = 4\cdot C_{y20}\cdot\sqrt{C_{y22}^2 + C_{x22}^2}\ ,$$

$$\beta = \mathrm{arctg}\,\left(\frac{C_{y22}}{C_{x22}}\right) + \pi\cdot q\ ,\quad q=0, \pm 1, \tag{21}$$

q should satisfy the conditions

$$\sin(\beta) = \frac{C_{y22}}{\sqrt{C_{y22}^2 + C_{x22}^2}}\ ,\quad \cos(\beta) = \frac{C_{x22}}{\sqrt{C_{y22}^2 + C_{x22}^2}}\ .$$

The spot has an ellipsoidal shape whose longer axis $2a = 2\cdot\delta_{\max}(\rho,\varphi)$ and shorter axis $2b = 2\cdot\delta_{\min}(\rho,\varphi)$, where (for ρ=1)

$$\delta_{\max} = 2\cdot\sqrt{C_{y22}^2 + C_{x22}^2 + 4\cdot C_{y20}^2 + 4\cdot\left|C_{y20}\right|\cdot\sqrt{C_{y22}^2 + C_{x22}^2}}\ , \tag{22}$$

$$\delta_{\min} = 2\cdot\sqrt{C_{y22}^2 + C_{x22}^2 + 4\cdot C_{y20}^2 - 4\cdot\left|C_{y20}\right|\cdot\sqrt{C_{y22}^2 + C_{x22}^2}}\ . \tag{23}$$

The longer axis is inclined relatively the vertical plane by an angle

$$\varphi_{\max} = \frac{\pi}{4} - \frac{\beta}{2}\quad \text{at } C_{y20} > 0\ , \tag{24}$$

or

$$\varphi_{\max} = -\frac{\pi}{4} - \frac{\beta}{2}\quad \text{at } C_{y20} < 0\ . \tag{25}$$

It is evident from (20) and (21), that ellipse's axes are oriented along coordinate axes (vertical and horizontal), when $C_{y22} \neq 0$ and $C_{x22} = 0$, and are inclined relatively the coordinate axes by an angle $\pm 45^0$, when $C_{y22} = 0$ and $C_{x22} \neq 0$. When $C_{y22} \neq 0$ and $C_{x22} \neq 0$, orientation of axes can be arbitrary (except the above mentioned). Position of astigmatic foci can be computed from the expression (20).

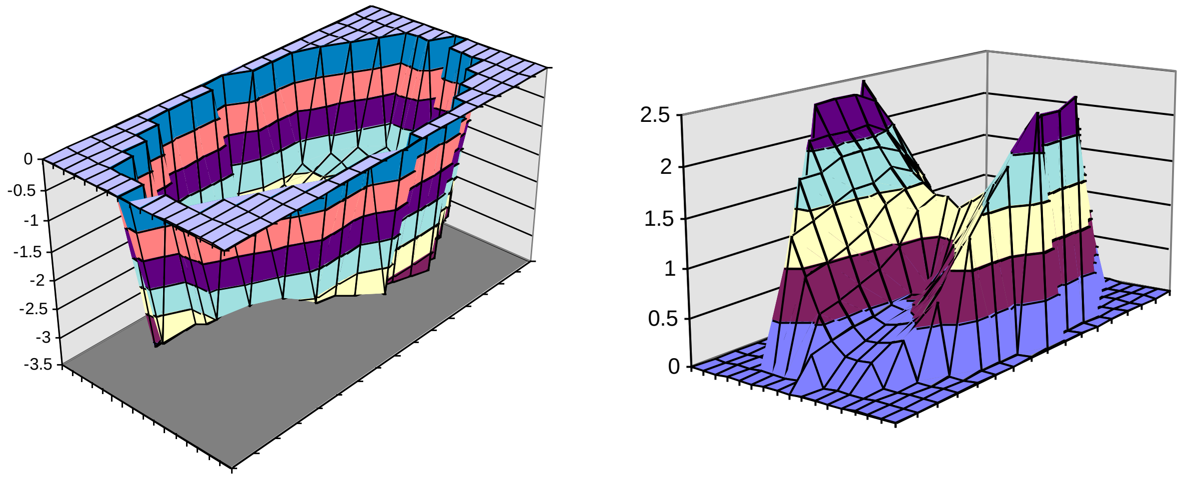
Astigmatic distance in diopters can be found as follows

$$\left|A'_s - A'_m\right| = \frac{8000\cdot n'}{f'\cdot D}\cdot\sqrt{C_{y22}^2 + C_{x22}^2}\ . \tag{26}$$

4. ANALYSIS OF EXPERIMENTAL DATA

The described technique was tested for two cases: myopic eye and hyperopic eye with astigmatism. Initial data (i. e., transverse aberrations) have been measured for each eye at 64 scan sites. Optimal values of approximation coefficients C_{ynm} and C_{xnm} were to be found. Eye's aberration parameters were computed, spot diagrams and wave front deformation maps being reconstructed as well, facilitating analysis of the shape and size of point spread function. Influence of selected types of aberrations could be analyzed in this way, as well as their combinations. Besides, spot diagrams could serve for sight acuity estimation before and after sight correction. Wave front deformation map is unambiguously correlated with the to-be-ablated cornea shape.

Transverse aberrations, recalculated into refraction map (distribution of focal power), are shown in fig. 1 for both cases. Some of the results of calculating C_{ynm} and C_{xnm} are included into tables 1 and 2.



a *b*

Fig. 1. Refraction maps [in diopters]: *a* - test 1; *b* - test 2

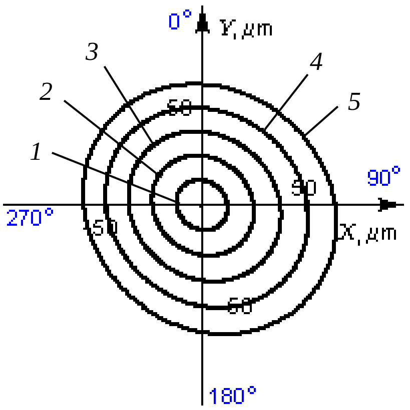
Table 1. Dependence of Zernike coefficients and RMSD on *n* and *m* (in µm) for test 1

	m=3, n=3	m=6, n=6	m_{opt}=3, n_{opt}=8	m=8, n=8	m=10, n=10
RMSD	4.06	2.60	2.53	2.70	6.40
C_{y20}	-15.66	-15.68	-15.68	-15.68	-15.31
C_{y40}	—	0.016	0.015	0.015	0.292
C_{y31}	0.511	-0.256	-0.215	-0.215	-0.205
C_{x31}	0.053	0.375	0.443	0.443	0.469
C_{y22}	0.278	0.225	0.210	0.210	0.189
C_{x22}	3.777	1.879	1.806	1.806	1.709

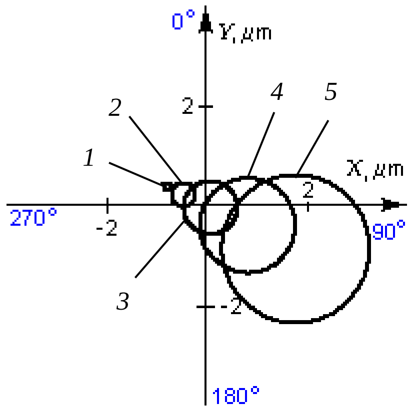
Table 2. Dependence of Zernike coefficients and RMSD on *n* and *m* (in µm) for test 2

	m=3, *n*=3	*m*=6, *n*=6	m_{opt}=2, n_{opt}=8	*m*=8, *n*=8	*m*=10, *n*=10
RMSD	8.10	4.50	4.41	4.78	7.57
C_{y20}	7.911	7.274	7.280	7.280	7.137
C_{y40}	—	0.469	0.460	0.460	0.222
C_{y31}	0.064	0.149	0.168	0.168	0.175
C_{x31}	-0.029	-0.099	-0.114	-0.114	-0.120
C_{y22}	-5.953	-2.770	-2.677	-2.677	-2.555
C_{x22}	-4.428	-1.964	-1.906	-1.906	-1.831

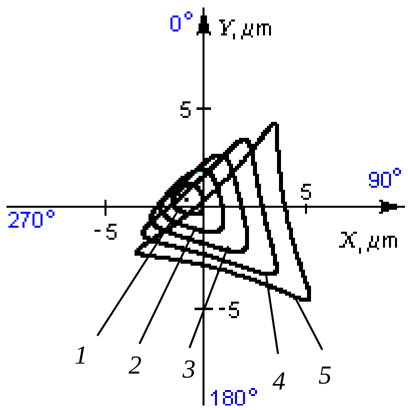
The procedure of search for optimal values of *m* and *n* is clear from the above tables: the combination of *m* and *n* must be found, for which RMSD is minimal. In our examples, they are: *n*=8, *m*=3 for test 1, and *n*=8, *m*=2 for test 2.
The values of Zernike coefficients, found in such a way, are substituted into expressions (8)…(26) for computation of correspondent aberration parameters and reconstruction of spot diagrams. Table 3 demonstrates the values of computed parameters, while figures 2 and 3 illustrate spot diagrams for some combinations of aberrations. Wave front deformation maps, reconstructed from computed Zernike coefficients, are shown in fig. 4.



a



b



c

Fig. 2. Retinal spot diagrams for test 1:
1 - ρ=0.2; *2* - ρ=0.4; *3* - ρ=0.6; *4* - ρ=0.8; *5* - ρ=1.0;
a - all primary aberrations (without tilts); *b* - 3rd order coma;
c - all primary aberrations (without tilts and defocusing)

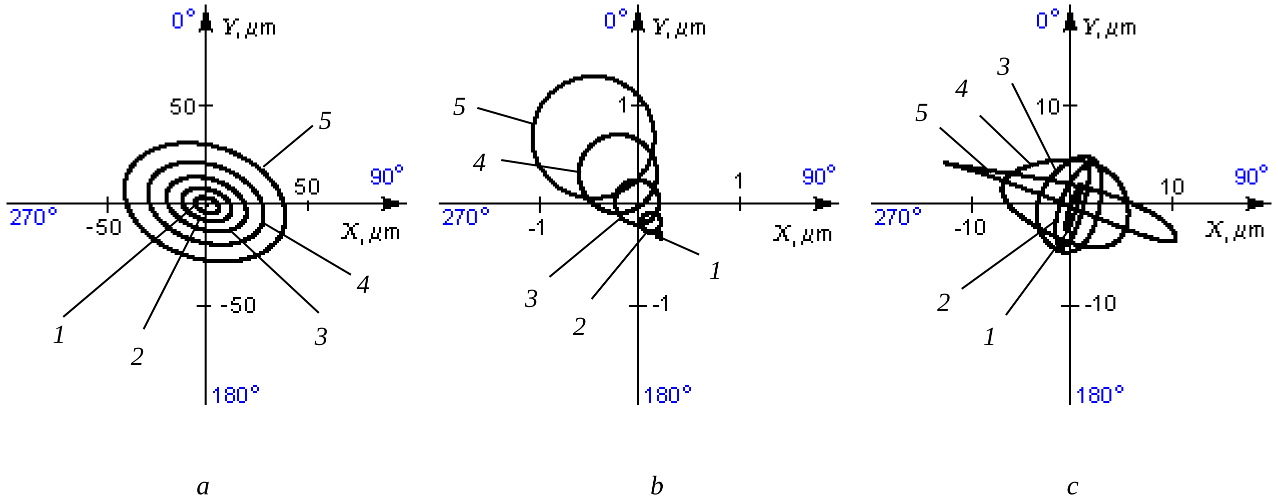
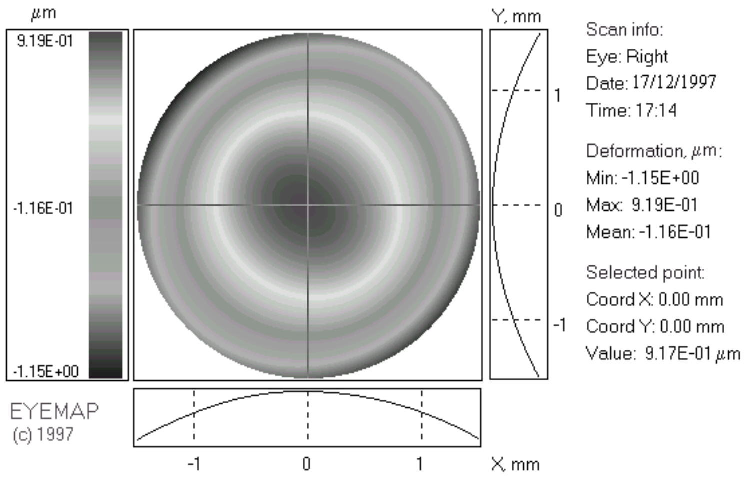


Fig. 3. Retinal spot diagrams for test 2:
1 -ρ=0.2; *2* - ρ=0.4; *3* - ρ=0.6; *4* - ρ=0.8; *5* - ρ=1.0;
a - all primary aberrations (without tilts); *b* - 3rd order coma;
c - all primary aberrations (without tilts and defocusing)

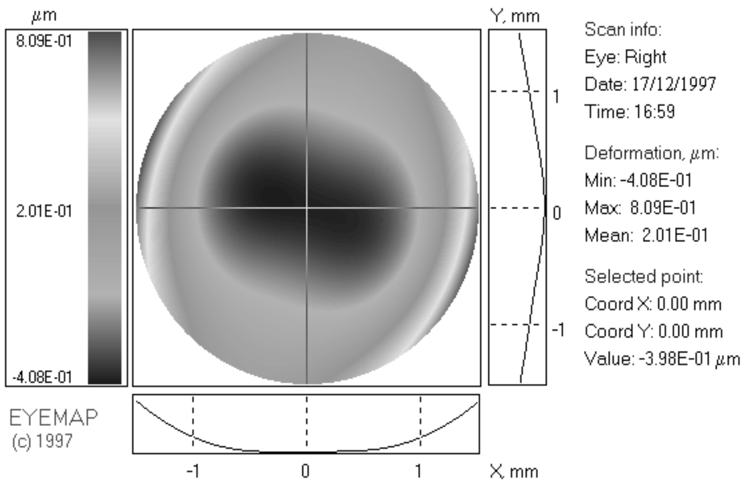
Table 3. Basic ophthalmic parameters of investigated eyes

	Ametropia	Spherical aberration		Coma				
	A [diopters]	$2\cdot r_s$ [μm]	θ [angular minutes]	$\delta l'_c$ [μm]	$\delta r'_c$ [μm]	α_c [degrees]	$\delta y'(0,0)$ [μm]	$\delta x'(0,0)$ [μm]
Test 1	-2.44	0.36	0.07	4.43	2.95	-25.89	0.43	-0.88
Test 2	+1.13	11.04	2.2	1.82	1.22	+124.16	-0.34	0.23

	Astigmatism			
	$2a$ [μm]	$2b$ [μm]	φ_{max} [degrees]	$\lvert Z'_s - Z'_m \rvert$ [diopters]
Test 1	132.7	118.16	-48.3	0.28
Test 2	71.38	45.08	-72.3	0.51



a



b

Fig. 4. Wave front deformation maps (usually in true colors): *a* - test 1; *b* - test 2

5. DISCUSSION AND CONCLUSIONS

The following results were got in measurements with myopic eye: ametropia- minus 2.44 diopters, 3rd order spherical aberration is negligible, there is also some astigmatism and coma. After ametropia correction, the shape of spot of light on retina will depend mainly on coma (fig. 2*c*). Its size is approximately примерно 10×10 µm. Therefore, sight acuity is 1.5 times worse than in normal eye (here we consider 6 µm spot on retina resulting in unity of sight acuity).

Hyperopic eye showed averaged hyperopia plus 1.13 diopters, value of 3rd order spherical aberration being essentially higher than in test 1. Coma and astigmatism are present with astigmatism being dominating. Therefore, after correction of ametropia retinal spot has an astigmatic shape (fig. 3*c*). Sight acuity at a plane of the longer axis of astigmatic ellipse (72.3 deg. inclined) is 1.7 times lower than in normal eye. Hence, astigmatism should not be neglected when reshaping the cornea.

Described methodology is quite effective for solving practical problems of optimal cornea reshaping in photorefractive sight correction. It is a good compromise between finite number of tested points when measuring transverse aberrations (limited by the time of measurement on moving eye) and required accuracy. The further freedom of development could include whether involving eye tracking to consider eye movements, or applying additional mathematics of the spline approximation type.

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